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c-Graded filiform Lie algebras

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Abstract. In this paper we generalize naturally graded filiform Lie algebras as well as filiform Lie algebras admitting a connected gradation of maximal length, by introducing the concept of c-graded complex filiform Lie algebras. We deal with the particular case of 3-graded filiform Lie algebras and we obtain their classification in arbitrary dimension. We finally show a link among derived algebras, graded filiform and rigid solvable Lie algebras.

Keywords: characteristically nilpotent Lie algebras, graded filiform Lie algebras, rigid solvable Lie algebras.

Mathematical subject classification: 17 B 30.

Introduction

Vergne introduced in [12] the concept of naturally graded filiform Lie algebras as those admitting a gradation associated with the lower central series. In that paper, she also classified them, up to isomorphism. Apart from that, several authors have studied algebras which admit a connected gradation of maximal length, this is, whose length is exactly the dimension of the algebra. So, Y. Hakimjanov started this study in [8], Reyes, in this Ph. D. Thesis [11] (later published in [3]), continued this research by giving an induction classification method and finally, Millionschikov in [10] gave the full list of these algebras (over an arbitrary field of zero characteristic).

The main goal of this paper is to introduce and study a kind of graded filiform Lie algebras, the c-graded ones, which generalizes both concepts recalled above. So, 1-graded filiform Lie algebras are naturally graded filiform Lie algebras and 2-graded filiform Lie algebras are the algebras by Hakimjanov. Although general results related c-graded filiform Lie algebras are obtained, these are

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particularized to establish the classification of 3-graded filiform Lie algebras in arbitrary dimension.

For several reasons, we think that this classification can suppose a step forward in getting the classification of Lie algebras in general. Indeed, by considering the usual gradation of the second space of Chevalley-Eilenberg's cohomology of the model filiform Lie algebra P_n, given by $H^2(P_n, P_n) = \oplus_{c\geq 0} H^2_{c+1}(P_n, P_n)$ and by taking into account that every filiform Lie algebra is isomorphic to $(P_n)_\psi$, where $\psi \in H^2(P_n, P_n)$ [12], it is possible to consider c-graded Lie algebras as algebras isomorphic to $(P_n)_{\psi_c}$, with $\psi_c \in H^2_{c+1}(P_n, P_n)$ (see [7]).

Therefore, the classification of c-graded Lie algebras allows, for one thing, to settle the classification of filiform Lie algebras of the type $(P_n)_{\psi_c+\psi_k}$ and, for another, to progress in the knowledge of the structure of $(P_n)_\psi$, because by considering the graduation, $\psi = \psi_c + \psi_{c+1} + \cdots + \psi_k$ then ψ_c and ψ_k have to be cocycles such that $(P_n)_{\psi_c}$ and $(P_n)_{\psi_k}$ are c-graded Lie algebras (see Remark 4.1).

Note also that c-graded Lie algebras are related with sill algebras introduced by Goze and Hakimjanov in [6], although they do not give any gradation for them.

Finally, by virtue of a result seen in [1], we obtain that for fixed $c \geq 2$, there exists $m = m(c) \in \mathbb{N}$ such that every c-graded Lie algebra of dimension $n \geq m$ is a derived algebra of a rigid solvable Lie algebra of dimension $n+1$.

1 Definitions and notations

In this paper, we will consider complex Lie algebras of finite dimension with laws denoted by [,].

In a Lie algebra $\mathcal{A}$, one can consider the lower central series: $C^1(\mathcal{A}) = \mathcal{A}$, $C^2(\mathcal{A}) = [\mathcal{A}, \mathcal{A}], \ldots, C^k(\mathcal{A}) = [C^{k-1}(\mathcal{A}), \mathcal{A}], \ldots$ It is said that $\mathcal{A}$ is filiform if $\dim C^k(\mathcal{A}) = n - k$, for $k \geq 2$, with $n = \dim \mathcal{A}$. Note that filiform Lie algebras are a subset of nilpotent Lie algebras.

As a consequence of Engel's Theorem, it is possible to obtain a basis $\{e_1, \ldots, e_n\}$ of every filiform Lie algebra such that,

$$[e_1, e_n] = 0, \quad [e_1, e_h] = e_{h+1} \quad (h = 2, \ldots, n-1), \quad [e_2, e_{n-1}] = 0.$$

Such a basis is called *adapted basis* and with respect to it, it is verified that $C^k(\mathcal{A}) = \langle e_{k+1}, \ldots, e_n \rangle$, $2 \leq k \leq n-1$.

According to a result by M. Vergne ([12]) there are two filiform Lie algebras

$\mathcal{A} = P_n$ whatever n is, and $\mathcal{A} = Q_n$ for $n \geq 4$ even, defined, respectively, by:

$$\begin{array}{llcll} P_n & : & [e_1, e_k] & = & e_{k+1} & 2 \leq k \leq n-1 \\ \\ Q_n & : & [e_1, e_k] & = & e_{k+1} & 2 \leq k \leq n-1 \\ & & [e_k, e_{n+1-k}] & = & (-1)^{k+1} e_n & 3 \leq k \leq \frac{n}{2} \\ & & [e_2, e_k] & = & e_{k+1} & 3 \leq k \leq n-2 \end{array}$$

which are both isomorphic to the graded algebra obtained when considering the lower central series, this is the gradation defined with respect to an adapted basis $\{e_1, \ldots, e_n\}$:

$$\begin{cases} \mathcal{H}_1 = \langle e_1, e_2 \rangle \\ \mathcal{H}_k = \langle e_{k+1} \rangle \qquad 2 \leq k \leq n-1. \end{cases}$$

So, $\mathcal{A} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_{n-1}$.

A Lie algebra $\mathcal{A}$ is said to be derived if there exists a Lie algebra $\mathcal{L}$ such that $\mathcal{A} = C^2(\mathcal{L})$.

2 c-graded Lie algebras

Definition 2.1. *A gradation $\{\mathcal{G}_i\}_{i \geq 1}$ of a Lie algebra $\mathcal{A}$ of dimension n is called a c-gradation if*

$$\mathcal{A} = \mathcal{G}_1 \oplus 0 \oplus \cdots \oplus 0 \oplus \mathcal{G}_c \oplus \ldots \oplus \mathcal{G}_{n+c-2},$$

where there are n one-dimensional nonzero homogeneous spaces, if $c > 2$ or

$$\mathcal{A} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \ldots \oplus \mathcal{G}_{n+c-2},$$

if $c = 2$.

Definition 2.2. *A filiform Lie algebra $\mathcal{A}$ is called c-graded for $c \geq 2$ if it admits a c-gradation.*

Note that for $c = 2$, we obtain a connected gradation, this is, a gradation without zero homogeneous elements. However if $c > 2$, we obtain a disconnected gradation with $c - 2$ zero homogeneous elements in it.

Proposition 2.1. *The filiform Lie algebra $\mathcal{A}$ of dimension n is c-graded if and only if there exists an adapted basis $B = \{e_1, \ldots, e_n\}$ of $\mathcal{A}$ such that:*

$$[e_h, e_k] = a_{h,k} e_{h+k+c-2}$$

where $a_{h,k} \in \mathbb{C}$, $h + k + c - 2 \leq n$, for all $e_h, e_k \in B$, with $1 < h, k < n$. In this case, the basis B is called a c-graded basis of $\mathcal{A}$.

Proof. Let consider the following c-gradation

$$\mathcal{A} = \mathcal{G}_1 \oplus 0 \oplus \cdots \oplus 0 \oplus \mathcal{G}_c \oplus \cdots \oplus \mathcal{G}_{n+c-2},$$

then we can take $z_1 \in \mathcal{G}_1$, $z_k \in \mathcal{G}_{k+c-2}$ for $2 \leq k \leq n$. As $\mathcal{A}$ is filiform, we have $[z_1, z_k] \neq 0$, for all k. So, by a suitable choice of $\alpha_l \in \mathbb{C}$, $1 \leq l \leq n$, we can consider $e_l = \alpha_l\, z_l$ and thus $\{e_1, \ldots, e_n\}$ is a c-graded basis satisfying brackets above.

Conversely, let $\mathcal{A}$ be a filiform Lie algebra whose brackets with respect to an adapted basis $\{e_1, \ldots, e_n\}$ are $[e_h, e_k] = a_{h,k} e_{h+k+c-2}$. Then we can consider the gradation in $\mathcal{A}$ defined by:

$$\begin{cases} \mathcal{G}_1 = \langle e_1 \rangle \\ \mathcal{G}_k = \langle e_{k-c+2} \rangle \qquad c \leq k \leq n+c-2. \end{cases}$$

We have $\mathcal{A} = \mathcal{G}_1 \oplus 0 \oplus \cdots \oplus 0 \oplus \mathcal{G}_c \oplus \ldots \oplus \mathcal{G}_{n+c-2}$. It completes the proof. $\square$

Lemma 2.2. *If $\mathcal{A}$ is a c-graded Lie algebra of dimension n, then*

$$2 \leq c \leq n-3.$$

Proof. Since $\mathcal{A}$ is c-graded filiform, if $\{e_1, \ldots, e_n\}$ is a c-graded basis, we have that:

$$[e_h, e_k] \in [\mathcal{A}, C^2(\mathcal{A})] = C^3(\mathcal{A})$$

for $1 < h, k \leq n$. So,

$$e_{h+k+c-2} \in \langle e_4, \ldots, e_n \rangle.$$

Therefore, $h+k+c-2 \leq n$. As $1 \neq h \neq k \neq 1$ then c will be maximun when $h = 2,\ k = 3$. $\square$

Remark 2.1. *According to previous definitions, the Lie algebras P_n, one for each dimension, are c-graded for all c. This is immediate if we consider $a_{h,k} = 0, \quad \forall h, k > 1$. From now on, these algebras will be called model Lie algebras.*

Moreover, by taking into account the following lemma, this condition characterizes the model algebras.

Lemma 2.3. *If a non-model Lie algebra is c_1-graded and c_2-graded, then $c_1 = c_2$.*

Proof. Let suppose $c_1 < c_2$. Let $B = \{e_1, \dots, e_n\}$ and $B' = \{e'_1, \dots, e'_n\}$ be c_1-graded and c_2-graded bases of $\mathcal{A}$ respectively. Since $\mathcal{A}$ is a non-model algebra, there exist $i, j \in \{2, \dots, n\}$, $i < j$ such that $[e_i, e_j] \neq 0$, and then $[e_i, e_j] = a_{i,j} e_{i+j+c_1-2}$.

By considering the adapted basis change between both base:

$$e_h = \sum_{k=1}^{n} C_{h,k} e'_k, \quad 1 \leq h \leq n,$$

we have by filiformity that $[e_1, e_h] = e_{h+1}$ for $2 \leq h \leq n-1$, therefore $C_{h,k} = 0$ for $k < h$, and $3 \leq h \leq n$. So, $e_n = C_{n,n} e'_n$ which implies $C_{n,n} \neq 0$. By a recurrence way it is easy to check that $C_{h,h} \neq 0$ for $3 \leq h \leq n$. So we have:

$$e_h = \sum_{k=1}^{n} C_{h,k} e'_k, \qquad h = 1, 2$$

$$e_h = \sum_{k \geq h} C_{h,k} e'_k, \quad C_{h,h} \neq 0, \quad h \geq 3$$

As B and B' are adapted, by considering $[e_2, e_{n-1}] = 0$, we deduce:

$$e_h = \sum_{k \geq h} C_{h,k} e'_k, \quad C_{h,h} \neq 0, \quad 1 \leq h \leq n.$$

As a consequence, we obtain that the matrix $(C_{h,k})$ of the change between two c-graded base is upper-triangular.

Note that:

$$[e_i, e_j] = a_{i,j} e_{i+j+c_1-2} = a_{i,j} \sum_{k \geq i+j+c_1-2} C_{i+j+c_1-2,k} e'_k =$$

$$= \left[\sum_{k \geq i} C_{i,k} e'_k, \sum_{h \geq j} C_{j,h} e'_h \right] = C_{i,i}\, C_{j,j}\, a'_{i,j}\, e'_{i+j+c_2-2} + \sum_{p > i+j+c_2-2} \lambda_p e'_p$$

where $\lambda_p \in \mathbb{C}$ and $a'_{i,j}$ are structure constants of $\mathcal{A}$ with respect to B'.

Therefore $a_{i,j} \neq 0$ and $i + j + c_1 - 2 < i + j + c_2 - 2$, which is contradictory. Finally we get a similar conclusion if we suppose that $c_2 < c_1$. $\square$

3 Rank of c-graded filiform Lie algebras

Let us recall that the rank of a nilpotent Lie algebra is the dimension of a maximal exterior torus of derivations. If the nilpotent Lie algebra is filiform, then its rank is strictly smaller than 3 (see [4]). Let $\mathcal{G}$ be a filiform Lie algebra. In [4] is also proved that $r(\mathcal{G}) = 2$ (where $r(\mathcal{G})$ denotes the rank of $\mathcal{G}$) if and only if $\mathcal{G} = P_n$ or $\mathcal{G} = Q_n$. Moreover, it can be checked that the torus of derivations if $\mathcal{G} = P_n$ is generated by:

$$g_1 = Id_{P_n} \quad g_2(e_1) = e_1, \quad g_2(e_i) = i\, e_i, \quad 2 \le i \le n$$

and if $\mathcal{G} = Q_n$, the torus is generated by:

$$h_1(z_1) = 0, \quad h_1(z_i) = z_i, \quad 2 \le i \le n-1 \quad h_1(z_n) = 2\, z_n$$

$$h_2(z_1) = z_1, \quad h_2(z_i) = (i-2)\, z_i, \quad 2 \le i \le n-1 \quad h_2(z_n) = (n-3)\, z_n$$

where $\{z_1, \dots, z_n\}$ is a basis of Q_n such that $z_1 = e_1 + e_2$, $z_i = e_i$, $i > 1$ and brackets are defined by $[z_1, z_i] = z_{i+1} \quad 2 \le i \le n-2$, $[z_1, z_{n-i+1}] = (-1)^{i+1} z_n$.

If $\mathcal{A}$ is a non model c-graded filiform Lie algebra with $c \ge 2$, then we can consider the following non nilpotent derivation:

$$d(e_1) = e_1, \quad d(e_2) = c\, e_2, \ \dots, \ d(e_n) = (n-2+c)\, e_n$$

where $\{e_1, \dots, e_n\}$ is a c-graded basis of $\mathcal{A}$. Moreover, as $\mathcal{A}$ is not isomorphic to Q_n, then $r(\mathcal{A}) = 1$.

Recall that characteristically nilpotent Lie algebras are those in which every derivation is nilpotent. For a general overview of these algebras, the reader can consult [5]. In [9] we have also proved that c-graded filiform Lie algebras are not characteristically nilpotent. So, as a consequence of these results, the following is true:

Proposition 3.1. *For $c \ge 2$, every c-graded filiform Lie algebra is derived from any solvable Lie algebra $\mathcal{L}$, of the form $\mathcal{L} = \mathcal{A} \oplus \langle U \rangle$, of dimension $n+1$, with $ad\ U = d$ ($ad\ U$ is the adjoint mapping $x \mapsto [U, x]$).* □

4 Structure of c-graded filiform Lie algebras

Let $\mathcal{A}$ be a c-graded filiform Lie algebra, with $c \ge 2$, and $B = \{e_1, \dots, e_n\}$ be a c-graded basis of $\mathcal{A}$. We will denote by $t_{q-1} \in \mathbb{C}$ the coefficient of e_{2q-1+c} in the brackets $[e_q, e_{q+1}]$, with $2 \leq q \leq [\frac{n-c+1}{2}]$, ($[x]$ denotes the integer part of x), that is:

$$\begin{array}{rcl} [e_2, e_3] & = & t_1\ e_{3+c} \\ [e_3, e_4] & = & t_2\ e_{5+c} \\ [e_4, e_5] & = & t_3\ e_{6+c} \\ & \vdots & \\ [e_{[\frac{n-c+1}{2}]}, e_{[\frac{n-c+1}{2}]+1}] & = & t_{[\frac{n-c+1}{2}]-1}\ e_{2[\frac{n-c+1}{2}]+c-1}. \end{array}$$

If we know these brackets, it is possible to determine the rest of brackets in $\mathcal{A}$ by considering, in a recurrent way, Jacobi identities associated with the elements e_1, e_q, e_p, for $q = [\frac{n-c+1}{2}], \ldots, 2$ and $p = q+1, \ldots, n+2-c-q$. These identities will be denoted by $J(e_1, e_q, e_p) = 0$.

Then, according to this notation the following result is verified:

Theorem 4.1. *The structure of $\mathcal{A}$ is the following:*

$$\begin{array}{lll} [e_1, e_q] & = e_{q+1} & 2 \leq q \leq n-1 \\ [e_q, e_{q+1}] & = t_{q-1}\ e_{2q+c-1} & 2 \leq q \leq [\dfrac{n-c+1}{2}] \\ [e_q, e_{q+2}] & = t_{q-1}\ e_{2q+c} & 2 \leq q \leq [\dfrac{n-c+1}{2}] \\ & & 2q+c \leq n \\ [e_q, e_p] & = \left(\displaystyle\sum_{l=0}^{[\frac{p-q-1}{2}]} (-1)^l \binom{p-1-q-l}{l} t_{q-1+l} \right) e_{q+p+c-2} & q+2 < p \leq n-q+2-c \\ & & 2 \leq q \leq [\dfrac{n-c+1}{2}] \end{array}$$

Moreover, as $\mathcal{A}$ is a Lie algebra, the rest of Jacobi identities are verified, that is:

$$J(e_q, e_p, e_r) = 0$$

with $2 \leq q < p < r \leq n-2c-1$ *and* $p+q+r-4+2c \leq n$. □

Remark 4.1. *By considering the second space $H^2(P_n, P_n)$ of Chevalley-Eilenberg's cohomology of the model filiform Lie algebra, it is easy to observe that the c-graded filiform Lie algebras are Lie algebras $(P_n)_{\psi_c}$ (see [7]), where $\psi_c \in H^2_{c+1}(P_n, P_n)$, this is, ψ_c belongs to one of the elements of the usual gradation $H^2(P_n, P_n) = \oplus_{c\geq 0} H^2_{c+1}(P_n, P_n)$.*

In order to classify c-graded filiform Lie algebras, $c \geq 2$, we will consider the general c-graded algebra of dimension n defined by Theorem 4.1, where t_i

are parameters in $\mathbb{C}$. The solution of the polynomial equations $P_{q,p,r} = 0$ (in $\mathbb{C}[t_1, \ldots, t_{[\frac{n-c+1}{2}]-1}]$), associated with Jacobi identities $J(e_q, e_p, e_r) = 0$, with $2 \leq q < p < r \leq n - 2c - 1$ and $p + q + r - 4 + 2c \leq n$, allows in the first place to determine the c-graded filiform Lie algebras and secondly to get their classification.

Taking it into account, we will denote by $\mathcal{A}_n^c(t_1, t_2, \ldots, t_{[\frac{n-c+1}{2}]-1})$ a Lie algebra whose law is defined as in Theorem 4.1, where t_i are the corresponding coefficients in $[e_{i+1}, e_{i+2}]$.

Proposition 4.2.

$$P_{q,p,r} = P_{q,p,r-1} - P_{q,p+1,r-1} - P_{q+1,p,r-1}$$

for $q < p < r$ *and* $p + q + r - 4 + 2c = n$.

Proof. Denoting the structure constants in $\mathcal{A}_n^c(t_1, t_2, \ldots, t_{[\frac{n-c+1}{2}]-1})$ by $a_{q,p}$, it is easily checked from Theorem 4.1, that:

$$a_{q,p} = a_{q+1,p} + a_{q,p+1}$$

for $q < p,\ q + p + c - 2 \leq n - 1$. As

$$P_{q,p,r} = a_{q,p}a_{q+p+c-2,r} - a_{q,r}a_{q+r+c-2,p} + a_{p,r}a_{p+r+c-2,q},$$

it completes the proof. $\square$

As a consequence of these relations among polynomial equations associated with Jacobi equations, it is possible (as the following theorem affirms) to obtain a smaller number of such polynomial equations, which constitute a generator system of the rest of equations.

Remark 4.2. *It is clear that if* $\{e_1, \ldots, e_n\}$ *is a* c*-graded basis of a* c*-graded Lie algebra,* $c \geq 2$, $\mathcal{A}_n^c(t_1, t_2, \ldots, t_{[\frac{n-c+1}{2}]-1})$*, then the ideal center is:*

$$Z(\mathcal{A}_n^c(t_1, t_2, \ldots, t_{[\frac{n-c+1}{2}]-1})) = \langle e_n \rangle.$$

According to Theorem 4.1, it can be proved that:

$$\mathcal{A}_n^c(t_1, t_2, \ldots, t_{[\frac{n-c+1}{2}]-1})/\langle e_n \rangle$$

is a c*-graded filiform Lie algebra of dimension* $n - 1$ *and* $\{e'_1 = e_1 + \langle e_n \rangle, e'_i = e_i + \langle e_n \rangle,\ 2 \leq i \leq n - 1\}$ *is a* c*-graded basis of it.*

Moreover,

$$\mathcal{A}_n^c(t_1, t_2, \dots, t_{[\frac{n-c+1}{2}]-1})/\langle e_n\rangle = \mathcal{A}_{n-1}^c(t_1, t_2, \dots, t_{[\frac{n-c}{2}]-1}).$$

This comes from the following fact: if we denote by $a_{q,p}$ the structure constants in the algebra $\mathcal{A}_n^c(t_1, t_2, \dots, t_{[\frac{n-c+1}{2}]-1})$ and $\bar{a}_{q,p}$ denote the structure constants in the algebra $\mathcal{A}_{n-1}^c(t_1, t_2, \dots, t_{[\frac{n-c}{2}]-1})$, it is verified that:

$$a_{q,p} = \bar{a}_{q,p}$$

for $p + q + c - 2 \le n - 1$. Then, if we denote by $\bar{P}_{q,p,r}$ the Jacobi polynomial equations in $\mathcal{A}_{n-1}^c(t_1, t_2, \dots, t_{[\frac{n-c}{2}]-1})$, we have that:

$$P_{q,p,r} = \bar{P}_{q,p,r}$$

for $q < p < r$ and $q + p + r - 4 + 2c \le n - 1$.

Recall that to obtain c-graded filiform Lie algebras, $c \ge 2$, of dimension n we need to solve the polynomial equations system $P_{q,p,r} = 0$, for $2 \le q < p < r \le n - 2c - 1$, $p + q + r - 4 + 2c \le n$. By using Remark 4.2, it is possible to determine these algebras by recurrence on the dimension. So, if we know the c-graded filiform Lie algebras of dimension $n - 1$, $\mathcal{A}_{n-1}^c(\lambda_1, \dots, \lambda_{[\frac{n-c}{2}]-1})$, then λ_i verify $P_{q,p,r} = 0$, for $q + p + r - 4 - 2c \le n - 1$. Hence the c-graded filiform Lie algebras $\mathcal{A} = \mathcal{A}_n^c(t_1, \dots, t_{[\frac{n-c+1}{2}]-1})$, of dimension n, are such that $t_i = \lambda_i$ for $1 \le i \le [\frac{n-c}{2}] - 1$ and t_i only must verify $P_{q,p,r} = 0$, for $q + p + r - 4 + 2c = n$.

Under these conditions, fixed the coefficients $\lambda_1, \dots, \lambda_{[\frac{n-c}{2}]-1}$ such that $\mathcal{A}_{n-1}^c(\lambda_1, \dots, \lambda_{[\frac{n-c}{2}]-1})$ is a c-graded filiform Lie algebra of dimension $n - 1$, the two following results are verified:

Theorem 4.3. *If $n - c$ is even, $\mathcal{A}_n^c(\lambda_1, \dots, \lambda_{[\frac{n-c}{2}]-1})$ is a c-graded filiform Lie algebra if and only if $\lambda_1, \dots, \lambda_{[\frac{n-c}{2}]-1}$ satisfy the following polynomial equations:*

- $J(e_{2+2k}, e_{\frac{n+1-2c}{2}-k}, e_{\frac{n+3-2c}{2}-k}) = 0$, *if n is odd, where $0 \le 3k \le \frac{n-2c-5}{2}$.*
- $J(e_{3+2k}, e_{\frac{n-2c}{2}-k}, e_{\frac{n-2c+2}{2}-k}) = 0$, *if n is even, where $0 \le 3k \le \frac{n-2c-8}{2}$.*

Theorem 4.4. *If $n-c$ is odd, $\mathcal{A}_n^c(\lambda_1, \dots, \lambda_{[\frac{n-c}{2}]-1}, t_{[\frac{n-c}{2}]})$ is a c-graded filiform Lie algebra if and only if $\lambda_1, \dots, \lambda_{[\frac{n-c}{2}]-1}, t_{[\frac{n-c}{2}]}$, satisfy the following polynomial equations:*

- $J(e_{2+2k}, e_{\frac{n+1-2c}{2}-k}, e_{\frac{n+3-2c}{2}-k}) = 0$, *if n is odd, where $0 \le 3k \le \frac{n-2c-5}{2}$.*

- $J(e_{3+2k}, e_{\frac{n-2c}{2}-k}, e_{\frac{n-2c+2}{2}-k}) = 0$, *if* n *is even, where* $0 \leq 3k \leq \frac{n-2c-8}{2}$.

To prove both results, according to Proposition 4.2, we have:

$$P_{q,p,r} = -P_{q,p+1,r} - P_{q+1,p,r}$$

for $q < p < r$, and $q + p + r - 4 + 2c = n$, since $P_{q,p,r-1} = 0$, because $q + p + r - 1 - 4 + 2c = n - 1$. Then, by recurrence on q, p, r, we obtain that any Jacobi polynomial equation is a linear combination of $P_{q,p,p+1}$ with $q + 2p + 1 - 4 + 2c = n$. □

Theorem 4.5. *Two* c*-graded filiform Lie algebras of dimension* n, $c \geq 2$, $\mathcal{A}_n^c(t_1, \ldots, t_{[\frac{n-c+1}{2}]-1})$ *and* $\mathcal{A}_n^c(t'_1, \ldots, t'_{[\frac{n-c+1}{2}]-1})$, *are isomorphic if and only if there exists a complex* $\lambda \neq 0$ *such that* $t'_h = \lambda\, t_h$ *for* $1 \leq h \leq [\frac{n-c+1}{2}] - 1$.

Proof. Let $\mathcal{A}_n^c(t_1, \ldots, t_{[\frac{n-c+1}{2}]-1})$ and $\mathcal{A}_n^c(\lambda t_1, \ldots, \lambda t_{[\frac{n-c+1}{2}]-1})$, $\lambda \neq 0$, be two filiform Lie algebras and $B = \{e_1, \ldots, e_n\}$, $B' = \{e'_1, \ldots, e'_n\}$, c-graded bases of them, respectively. The basis change $e'_1 = e_1$, $e'_i = \lambda e_i$, $2 \leq i \leq n$ proves that these two algebras are isomorphic.

Conversely, if we consider two c-graded bases B and B' of the algebras $\mathcal{A}_n^c(t_1, \ldots, t_{[\frac{n-c+1}{2}]-1})$ and $\mathcal{A}_n^c(t'_1, \ldots, t'_{[\frac{n-c+1}{2}]-1})$ respectively, satisfying the hypothesis of the theorem, we assume that there exists a basis change:

$$e'_h = \sum_{k \geq h} C_{h,k} e_k, \quad C_{k,k} \neq 0, \quad 1 \leq h \leq n.$$

Starting from $[e'_1, e'_k] = e'_{k+1}$, $2 \leq k \leq n-1$, we obtain that:

$$C_{k,k} = (C_{1,1})^{n-k} C_{2,2}, \quad 3 \leq k \leq n.$$

Moreover, if we denote by $a'_{q,p}$ the structure constants in $\mathcal{A}_n^c(t'_1, \ldots, t'_{[\frac{n-c+1}{2}]-1})$ with respect to B' and by considering $[e'_q, e'_p]$ the ones with respect to B, we conclude that:

$$a_{q,p} C_{p,p} C_{q,q} = a'_{q,p} C_{p+q-2+c,p+q+c-2}$$

for $1 \leq q < p \leq n$, $\quad p + q + c - 2 \leq n$. Consequently,

$$t'_{q-1} = a'_{q,q+1} = \frac{C_{q,q} C_{q+1,q+1}}{C_{2q+c-1,2q+c-1}} t_{q-1} = C_{2,2}(C_{1,1})^{n+c-1} t_{q-1}$$

for $2 \leq q \leq [\frac{n-c+1}{2}]$. Hence $\lambda = C_{2,2}(C_{1,1})^{n+c-1} \neq 0$ verifies required conditions. □

Remark 4.3. *Note that, if we have two c-graded filiform Lie algebras $\mathcal{A}_n^c$, $\mathcal{A}'^c_n$ such that $\mathcal{A}_n^c/Z(\mathcal{A}_n^c)$ is not isomorphic to $\mathcal{A}'^c_n/Z(\mathcal{A}'^c_n)$, then $\mathcal{A}_n^c$ is not isomorphic to $\mathcal{A}'^c_n$.*

5 Classification of 3-graded filiform Lie algebras

By using the notation of Theorem 4.1 and by taking into consideration the previous theorem, we have that every 3-graded filiform Lie algebra is isomorphic to one of the following Lie algebras. Note that for dimension less than 6, the unique 3-graded Lie algebra in each dimension is the model algebra.

- In $\dim \mathcal{A} = 6$: $\mathbf{g}_6^1 := \mathcal{A}_6^3(0)$, $\mathbf{g}_6^2 := \mathcal{A}_6^3(1)$.
- In $\dim \mathcal{A} = 7$: $\mathbf{g}_7^1 := \mathcal{A}_7^3(0)$, $\mathbf{g}_7^2 := \mathcal{A}_7^3(1)$.
- In $\dim \mathcal{A} = 8$: $\mathbf{g}_8^1 := \mathcal{A}_8^3(0,0)$, $\mathbf{g}_8^2 := \mathcal{A}_8^3(\lambda,1)$, $\mathbf{g}_8^3 := \mathcal{A}_8^3(1,0)$.
- In $\dim \mathcal{A} = 9$: $\mathbf{g}_9^1 := \mathcal{A}_9^3(0,0)$, $\mathbf{g}_9^2 := \mathcal{A}_9^3(\lambda,1)$, $\mathbf{g}_9^3 := \mathcal{A}_9^3(1,0)$.
- In $\dim \mathcal{A} = 10$: $\mathbf{g}_{10}^1 := \mathcal{A}_{10}^3(0,0,0)$, $\mathbf{g}_{10}^2 := \mathcal{A}_{10}^3(\lambda,1,0)$,
  $\mathbf{g}_{10}^3 := \mathcal{A}_{10}^3(1,0,0)$, $\mathbf{g}_{10}^4 := \mathcal{A}_{10}^3(\alpha,\beta,1)$ with $\lambda,\ \alpha,\ \beta\ \in \mathbb{C}$.
- In $\dim \mathcal{A} = 11$: $\mathbf{g}_{11}^1 := \mathcal{A}_{11}^3(0,0,0)$, $\mathbf{g}_{11}^2 := \mathcal{A}_{11}^3(1,0,0)$,
  $\mathbf{g}_{11}^3 := \mathcal{A}_{11}^3(\frac{4\lambda^2-3\lambda}{3},\lambda,1)$ with $\lambda \in \mathbb{C}$.
- In $\dim \mathcal{A} = 12$: $\mathbf{g}_{12}^1 := \mathcal{A}_{12}^3(0,0,0,0)$, $\mathbf{g}_{12}^2 := \mathcal{A}_{12}^3(1,0,0,\lambda)$,
  $\mathbf{g}_{12}^3 := \mathcal{A}_{12}^3(0,0,0,1)$, $\mathbf{g}_{12}^4 := \mathcal{A}_{12}^3(\frac{4\lambda^2-3\lambda}{3},\lambda,1,\mu)$ with $\lambda,\ \mu\ \in \mathbb{C}$.
- In $\dim \mathcal{A} = 13$: $\mathbf{g}_{13}^1 := \mathcal{A}_{13}^3(0,0,0,0)$, $\mathbf{g}_{13}^2 := \mathcal{A}_{13}^3(1,0,0,0)$,
  $\mathbf{g}_{13}^3 := \mathcal{A}_{13}^3(0,0,0,1)$, $\mathbf{g}_{13}^4 := \mathcal{A}_{13}^3(\frac{4\lambda^2-3\lambda}{3},\lambda,1,\frac{5\lambda-10}{4\lambda^2-5\lambda-4})$ for $\lambda \in \mathbb{C}$ such that $4\lambda^2-5\lambda-4 \neq 0$.
- In $\dim \mathcal{A} = 14$: $\mathbf{g}_{14}^1 := \mathcal{A}_{14}^3(0,0,0,0,0)$,
  $\mathbf{g}_{14}^2 := \mathcal{A}_{14}^3(0,0,0,0,1)$, $\mathbf{g}_{14}^3 := \mathcal{A}_{14}^3(1,0,0,0,\alpha)$,
  $\mathbf{g}_{14}^4 := \mathcal{A}_{14}^3(0,0,0,1,\alpha)$ $\alpha \in \mathbb{C}$
  $\mathbf{g}_{14}^5 := \mathcal{A}_{14}^3(\frac{4\lambda^2-3\lambda}{3},\lambda,1,\frac{5\lambda-10}{4\lambda^2-5\lambda-4},\frac{15\lambda-40}{8\lambda^3-6\lambda^2-13\lambda-4})$ for $\lambda \in \mathbb{C}$ such that $4\lambda^2-5\lambda-4 \neq 0$ and $8\lambda^3-6\lambda^2-13\lambda-4 \neq 0$.

- In $\dim \mathcal{A} = 15$: $\mathbf{g}_{15}^1 := \mathcal{A}_{15}^3(0, 0, 0, 0, 0)$,
  $\mathbf{g}_{15}^2 := \mathcal{A}_{15}^3(0, 0, 0, 0, 1)$, $\mathbf{g}_{15}^3 := \mathcal{A}_{15}^3(1, 0, 0, 0, 0)$,
  $\mathbf{g}_{15}^4 := \mathcal{A}_{15}^3(0, 0, 0, 1, 4)$,
  $\mathbf{g}_{15}^5 := \mathcal{A}_{15}^3(42, 6, 1, \frac{2}{11}, \frac{5}{11\cdot 13})$, $\mathbf{g}_{15}^6 := \mathcal{A}_{15}^3(\frac{10}{3}, 2, 1, 0, -1)$,
  $\mathbf{g}_{15}^7 := \mathcal{A}_{15}^3(\frac{21+7\sqrt{11}}{12}, \frac{5+\sqrt{11}}{4}, 1, \frac{-3+\sqrt{11}}{-1+\sqrt{11}}, \frac{-17+3\sqrt{11}}{2+3\sqrt{11}})$,
  $\mathbf{g}_{15}^8 := \mathcal{A}_{15}^3(\frac{21-7\sqrt{11}}{12}, \frac{5-\sqrt{11}}{4}, 1, \frac{3+\sqrt{11}}{1+\sqrt{11}}, \frac{17+3\sqrt{11}}{-2+3\sqrt{11}})$.
- In $\dim \mathcal{A} = 16$: $\mathbf{g}_{16}^1 := \mathcal{A}_{16}^3(0, 0, 0, 0, 0, 0)$,
  $\mathbf{g}_{16}^2 := \mathcal{A}_{16}^3(0, 0, 0, 0, 0, 1)$, $\mathbf{g}_{16}^3 := \mathcal{A}_{16}^3(0, 0, 0, 0, 1, \lambda)$,
  $\mathbf{g}_{16}^4 := \mathcal{A}_{16}^3(42, 6, 1, \frac{2}{11}, \frac{5}{11\cdot 13}, \frac{1}{11\cdot 13})$, $\mathbf{g}_{16}^5 := \mathcal{A}_{16}^3(1, 0, 0, 0, 0, \lambda)$,
  $\mathbf{g}_{16}^6 := \mathcal{A}_{16}^3(0, 0, 0, 1, 4, 25)$, $\mathbf{g}_{16}^7 := \mathcal{A}_{16}^3(\frac{10}{3}, 2, 1, 0, -1, \frac{-5}{3})$,
  $\mathbf{g}_{16}^8 := \mathcal{A}_{16}^3(\frac{21+7\sqrt{11}}{12}, \frac{5+\sqrt{11}}{4}, 1, \frac{-3+\sqrt{11}}{-1+\sqrt{11}}, \frac{-17+3\sqrt{11}}{2+3\sqrt{11}}, 6\frac{-52+17\sqrt{11}}{-31+\sqrt{11}})$,
  $\mathbf{g}_{16}^9 := \mathcal{A}_{16}^3(\frac{21-7\sqrt{11}}{12}, \frac{5-\sqrt{11}}{4}, 1, \frac{3+\sqrt{11}}{1+\sqrt{11}}, \frac{17+3\sqrt{11}}{-2+3\sqrt{11}}, 6\frac{52+17\sqrt{11}}{31-\sqrt{11}})$, $\lambda \in \mathbb{C}$.

We show now in detail some examples of c-graded Lie algebras which allow to check the classification above indicated. We use dimension 6 and 13 due to in each of these cases the computations are different. Later, we will continue with the classification for $\dim \mathcal{A} \geq 17$.

Example 5.1. *Let $\mathcal{A}$ be a 3-graded Lie algebra of dimension 6.*

In this case, according to Theorem 4.1 we have an one-parametric family of algebras:

$$\begin{array}{llcll} \mathcal{A}_6^3(t_1): & [e_1, e_k] & = & e_{k+1} & \textit{for} \quad 2 \leq k \leq 5 \\ & [e_2, e_3] & = & t_1\, e_6 & \end{array}$$

where the parameter $t_1 \in \mathbb{C}$. So, according to Theorem 4.5, we distinguish:

- *If $t_1 = 0$, we obtain the model algebra L_6,*

$$\mathcal{A}_6^3(0): \quad [e_1, e_k] \; = \; e_{k+1} \quad \textit{for} \quad 2 \leq k \leq 5.$$

- *If $t_1 \neq 0$, all the algebras $\mathcal{A}_6^3(t_1)$ with $t_1 \neq 0$ are isomorphic to:*

$$\begin{array}{llcll} \mathcal{A}_6^3(1): & [e_1, e_k] & = & e_{k+1} & \textit{for} \quad 2 \leq k \leq 5 \\ & [e_2, e_3] & = & e_6. & \end{array}$$

Then, a 3*-graded filiform Lie algebra of dimension* 6 *is isomorphic to* $\mathcal{A}_6^3(0)$ *or* $\mathcal{A}_6^3(1)$.

The classification of 3-graded Lie algebras up dimension 12 can be obtained in a similar way.

To classify 3-graded Lie algebras of dimension greater or equal than 13 we use an inductive method based on Theorems 4.1, 4.3, 4.4 and 4.5 and Remarks 4.2 and 4.3.

Example 5.2. *We start from the four-parametric family* $\mathcal{A}_{12}^3(t_1, t_2, t_3, t_4)$ *of algebras in dimension* 12*, (see Theorem 4.1), where the parameters* t_1, t_2, t_3 *and* t_4 *have to verify the condition* $p1 = P_{2,3,4}$*:*

$$p1 = 3t_2t_3 + 3t_1t_3 - 4t_2^2 = 0,$$

and we obtain that a 3*-graded filiform Lie algebra of dimension* 12 *is isomorphic to one of the following:* $\mathcal{A}_{12}^3(0,0,0,0)$, $\mathcal{A}_{12}^3(0,0,0,1)$, $\mathcal{A}_{12}^3(1,0,0,\alpha)$ *or* $\mathcal{A}_{12}^3(\frac{4\lambda^2-3\lambda}{3}, \lambda, 1, \mu)$*, where* $\alpha, \lambda, \mu \in \mathbb{C}$.

Now, for dimension 13*, starting from the four-parametric family of algebras:*

$$\mathcal{A}_{13}^3(t_1,t_2,t_3,t_4): \quad \begin{array}{lcl} [e_1, e_k] & = & e_{k+1} \quad \textit{for} \quad 2 \le k \le 12 \\ [e_2, e_{10}] & = & (t_1 - 6t_2 + 10t_3 - 4t_4)\ e_{13} \\ [e_2, e_9] & = & (t_1 - 5t_2 + 6t_3 - t_4)\ e_{12} \\ [e_2, e_8] & = & (t_1 - 4t_2 + 3t_3)\ e_{11} \\ [e_2, e_7] & = & (t_1 - 3t_2 + t_3)\ e_{10} \\ [e_2, e_6] & = & (t_1 - 2t_2)\ e_9 \\ [e_2, e_5] & = & (t_1 - t_2)\ e_8 \\ [e_2, e_4] & = & t_1\ e_7 \\ [e_2, e_3] & = & t_1\ e_6 \\ [e_3, e_9] & = & (t_2 - 4t_3 + 3t_4)\ e_{13} \\ [e_3, e_8] & = & (t_2 - 3t_3 + t_4)\ e_{12} \\ [e_3, e_7] & = & (t_2 - 2t_3)\ e_{11} \\ [e_3, e_6] & = & (t_2 - t_3)\ e_{10} \\ [e_3, e_5] & = & t_2\ e_9 \\ [e_3, e_4] & = & t_2\ e_8 \\ [e_4, e_8] & = & (t_3 - 2t_4)\ e_{13} \\ [e_4, e_7] & = & (t_3 - t_4)\ e_{12} \\ [e_4, e_6] & = & t_3\ e_{11} \\ [e_4, e_5] & = & t_3\ e_{10} \\ [e_5, e_7] & = & t_4\ e_{13} \\ [e_5, e_6] & = & t_4\ e_{12} \end{array}$$

where the parameters t_1, t_2, t_3 *and* t_4 *have to verify conditions* $p1 = 0$ *(see Remark 4.2) and* $p2 = P^{13=0}_{2,4,5}$, *which is reduced to:*

$$p2 = -5t_2t_3 + 10t_3^2 - 4t_3t_4 + 3t_1t_4 - 2t_2t_4 = 0 \tag{1}$$

by taking into account Theorem 4.3.

Hence, from classification for dimension 12 *and Remark 4.3, we conclude:*

- *From* $\langle(0,0,0,0)\rangle$ *we obtain the model algebra* $\mathcal{A}^3_{13}(0,0,0,0)$, *because* $(0,0,0,0)$ *verifies (1).*
- *From* $\cup_{\lambda\in\mathbb{C}}\langle(1,0,0,\lambda)\rangle$ *we obtain* $\mathcal{A}^3_{13}(1,0,0,0)$, *since* $\lambda = 0$ *is necessary for* $(1,0,0,\lambda)$ *to verify (1).*
- *From* $\langle(0,0,0,1)\rangle$ *we have* $\mathcal{A}^3_{13}(0,0,0,1)$, *because* $(0,0,0,1)$ *verifies(1).*
- *From* $\cup_{\lambda,\mu\in\mathbb{C}}\langle(\frac{4\lambda^2-3\lambda}{3},\lambda,1,\mu)\rangle$ *we obtain* $\mathcal{A}^3_{13}(\frac{4\lambda^2-3\lambda}{3},\lambda,1,\frac{5\lambda-10}{4\lambda^2-5\lambda-4})$, *since* $\mu = \frac{5\lambda-10}{4\lambda^2-5\lambda-4}$, *for* $\lambda \in \mathbb{C}$ *with* $4\lambda^2 - 5\lambda - 4 \neq 0$ *is necessary for* $(\frac{4\lambda^2-3\lambda}{3},\lambda,1,\mu)$ *to verify (1).*

Next, we prove that for dim $\mathcal{A} \geq 17$, only five 3-graded filiform Lie algebras (up to isomorphism) are obtained in the case of even dimension, whereas, four algebras and two one-parametric families of 3-graded filiform Lie algebras (up to isomorphism) are obtained in the case of odd dimension.

Theorem 5.1. *Let* $\mathcal{A}$ *be a* 3*-graded filiform Lie algebra of dimension* n, *with* $n \geq 17$. *Then they are verified:*

- *If* n *is odd, the Lie algebra* $\mathcal{A}^3_n(t_1,\ldots,t_{\frac{n-5}{2}})$ *is isomorphic to one of the following algebras:*
  - $\mathcal{A}^3_n(0,0,\ldots,0)$
  - $\mathcal{A}^3_n(1,0,\ldots,0)$
  - $\mathcal{A}^3_n(0,\ldots,0,1)$
  - $\mathcal{A}^3_n(0,\ldots,0,1,\frac{(n-3)(n-7)}{24})$
  - $\mathcal{A}^3_n(\lambda_1,\ldots,\lambda_{\frac{n-5}{2}})$, *where* $\lambda_{k+1} = \frac{k+1}{2(2k+5)}\lambda_k$ *for* $k \geq 1$ *and* $\lambda_1 = 42$.
- *If* n *is even, the Lie algebra* $\mathcal{A}^3_n(t_1,\ldots,t_{\frac{n-4}{2}})$ *is isomorphic to one of the following algebras:*

– $\mathcal{A}_n^3(0, 0, \ldots, 0)$
– $\mathcal{A}_n^3(1, 0, \ldots, 0, \lambda)$ *with* $\lambda \in \mathbb{C}$
– $\mathcal{A}_n^3(0, \ldots, 1, \lambda)$ *with* $\lambda \in \mathbb{C}$
– $\mathcal{A}_n^3(0, \ldots, 0, 1)$
– $\mathcal{A}_n^3(0, \ldots, 0, 1, \frac{(n-4)(n-8)}{24}, \frac{(n-6)^2(n-8)(n-4)}{3 \cdot 32 \cdot 4})$
– $\mathcal{A}_n^3(\lambda_1, \ldots, \lambda_{\frac{n-4}{2}})$, *where* $\lambda_{k+1} = \frac{k+1}{2(2k+5)}\lambda_k$ *for* $k \geq 1$ *and* $\lambda_1 = 42$.

Proof. We proceed by induction on $n = \dim \mathcal{A}$. Note that some details of the induction will be omitted, due to the proof could be lengthy.

- For $\dim \mathcal{A} = 17$ we obtain:
  $\mathbf{g}_{17}^1 := \mathcal{A}_{17}^3(0, 0, 0, 0, 0, 0)$, $\mathbf{g}_{17}^2 := \mathcal{A}_{17}^3(0, 0, 0, 0, 0, 1)$,
  $\mathbf{g}_{17}^3 := \mathcal{A}_{17}^3(0, 0, 0, 0, 1, \frac{35}{6})$ $\mathbf{g}_{17}^4 := \mathcal{A}_{17}^3(1, 0, 0, 0, 0, 0)$,
  $\mathbf{g}_{17}^5 := \mathcal{A}_{17}^3(42, 6, 1, \frac{2}{11}, \frac{5}{11 \cdot 13}, \frac{1}{11 \cdot 13})$.
- For $\dim \mathcal{A} = 18$:
  $\mathbf{g}_{18}^1 := \mathcal{A}_{18}^3(0, 0, 0, 0, 0, 0, 0)$, $\mathbf{g}_{18}^2 := \mathcal{A}_{18}^3(0, 0, 0, 0, 0, 0, 1)$,
  $\mathbf{g}_{18}^3 := \mathcal{A}_{18}^3(0, 0, 0, 0, 0, 1, \lambda)$, $\mathbf{g}_{18}^4 := \mathcal{A}_{18}^3(0, 0, 0, 0, 1, \frac{35}{6}, \frac{105}{2})$,
  $\mathbf{g}_{18}^5 := \mathcal{A}_{18}^3(42, 6, 1, \frac{2}{11}, \frac{5}{11 \cdot 13}, \frac{1}{11 \cdot 13}, \frac{7}{11 \cdot 13 \cdot 17 \cdot 2})$,
  $\mathbf{g}_{18}^6 := \mathcal{A}_{18}^3(1, 0, 0, 0, 0, 0, \lambda)$, $\lambda \in \mathbb{C}$.

Let suppose the result proved for $\dim \mathcal{A} = n - 1$. We will show that it is also true for $\dim \mathcal{A} = n$. Indeed:

- If n es odd. By using Theorem 4.3 we have that the parameters $t_1, \ldots, t_{\frac{n-5}{2}}$ of any 3-graded filiform Lie algebra $\mathcal{A}_n^3(t_1, \ldots, t_{\frac{n-5}{2}})$ have to satisfy:

$$J(e_{2+2r}, e_{\frac{n-5}{2}-r}, e_{\frac{n-3}{2}-r}) = 0$$

where $0 \leq 3r \leq \frac{n-11}{2}$. From Theorem 4.1, it is equivalent to:

$$\begin{aligned} &- t_{\frac{n-7-2r}{2}}[e_{2r+2}, e_{n-2r-3}] - \\ &- \sum_{l=0}^{[\frac{n-6r-9}{4}]} (-1)^l \binom{\frac{n-6r-9}{2} - l}{l} t_{2r+1+l}[e_{\frac{n+2r+3}{2}}, e_{\frac{n-5-2r}{2}}] + \\ &+ \sum_{l=0}^{[\frac{n-6r-11}{4}]} (-1)^l \binom{\frac{n-6r-11}{2} - l}{l} t_{2r+1+l}[e_{\frac{n+2r+1}{2}}, e_{\frac{n-3-2r}{2}}] = 0. \end{aligned} \tag{2}$$

By hypothesis, the algebra $\mathcal{A}^3_{n-1}(t_1, \dots, t_{\frac{n-5}{2}})$ is isomorphic to one of the following: $\mathcal{A}^3_{n-1}(0, \dots, 0)$, $\mathcal{A}^3_{n-1}(1, \dots, 0, \lambda)$, $\mathcal{A}^3_{n-1}(0, \dots, 0, 1)$, $\mathcal{A}^3_{n-1}(0, \dots, 0, 1, \lambda)$, $\mathcal{A}^3_{n-1}(0, \dots, 0, 1, \frac{(n-5)(n-9)}{24}, \frac{(n-7)^2(n-9)(n-5)}{32\cdot 4\cdot 3})$ or $\mathcal{A}^3_{n-1}(\lambda_1, \dots, \lambda_{\frac{n-5}{2}})$, where $\lambda_{k+1} = \frac{k+1}{4k+10}\lambda_k$, for $k \geq 1$ and $\lambda_1 = 42$. Then:

– From the first four algebras previously mentioned, we obtain, respectively: $\mathcal{A}^3_n(0, \dots, 0)$, $\mathcal{A}^3_n(0, \dots, 0, 1)$, $\mathcal{A}^3_n(1, 0, \dots, 0)$ and $\mathcal{A}^3_n(0, \dots, 0, 1, \frac{(n-3)(n-7)}{24})$.

– From $\mathcal{A}^3_{n-1}\big(0, \dots, 0, 1, \frac{(n-5)(n-9)}{24}, \frac{(n-7)^2(n-9)(n-5)}{32\cdot 4\cdot 3}\big)$ we do not obtain any Lie algebra, because

$$\left(0, \dots, 0, 1, \frac{(n-5)(n-9)}{24}, \frac{(n-7)^2(n-9)(n-5)}{32\cdot 4\cdot 3}\right)$$

does not satisfy $J(e_4, e_{\frac{n-7}{2}}, e_{\frac{n-5}{2}}) = 0$.

– From $\mathcal{A}^3_{n-1}(\lambda_1, \dots, \lambda_{\frac{n-5}{2}})$, where $\lambda_{k+1} = \frac{k+1}{2(2k+5)}\lambda_k$ for $k \geq 1$ and $\lambda_1 = 42$, we obtain $\mathcal{A}^3_n(\lambda_1, \dots, \lambda_{\frac{n-5}{2}})$, since it is isomorphic to a finite quotient algebra of the infinite dimensional Witt Lie algebra:

$$W_\infty : [x_i, x_j] = (j-i)x_{i+j} \text{ for } 1 \leq i, j$$

defined by:

$$W_\infty / \langle x_2, x_{n+2}, \dots \rangle.$$

So, with respect to the basis $\{x_1, x_3, x_4, \dots, x_{n+1}\}$, the quotient algebra is defined by $[x_i, x_j] = (j-i)x_{i+j}$ for $i + j \leq n+1$. And the isomorphism is defined by $e_1 = x_1$, $e_i = 6\,(i-1)!\,420\, x_{i+1}$ for $i \geq 2$.

• If n is even, by taking into account the Theorem 4.4 we have that the parameters $t_1, \dots, t_{\frac{n-4}{2}}$ of any 3-graded filiform Lie algebra $\mathcal{A}^3_n(t_1, \dots, t_{\frac{n-4}{2}})$ have to satisfy:

$$J(e_{3+2r}, e_{\frac{n-6}{2}-r}, e_{\frac{n-4}{2}-r}) = 0$$

where $0 \leq 3r \leq \frac{n-14}{2}$. Then,

$$
\begin{aligned}
& t_{n-\frac{n-8-2r}{2}}[e_{n-4-2r}, e_{3+2r}] - \\
& - \sum_{l=0}^{[\frac{n-6r-12}{4}]} (-1)^l \binom{\frac{n-6r-12}{2} - l}{l} t_{2r+2+l}[e_{\frac{n+}{2}+r}, e_{\frac{n-6}{2}-r}] + \\
& + \sum_{l=0}^{[\frac{n-6r-14}{4}]} (-1)^l \binom{\frac{n-6r-14}{2} - l}{l} t_{2r+2+l}[e_{\frac{n+2}{2}+r}, e_{\frac{n-4}{2}-r}] = 0.
\end{aligned}
\tag{3}
$$

By hypothesis, the algebra $\mathcal{A}^3_{n-1}(t_1, \ldots, t_{\frac{n-6}{2}})$ is isomorphic to one of the following algebras:

$$
\mathcal{A}^3_{n-1}(0, \ldots, 0), \quad \mathcal{A}^3_{n-1}(1, \ldots, 0), \quad \mathcal{A}^3_{n-1}(0, \ldots, 0, 1),
$$
$$
\mathcal{A}^3_{n-1}\Big(0, \ldots, 0, 1, \frac{(n-4)(n-8)}{24}\Big) \quad \text{or} \quad \mathcal{A}^2_{n-1}(\lambda_1, \ldots, \lambda_{\frac{n-5}{2}}),
$$

where $\lambda_{k+1} = \frac{k+1}{4k+10}\lambda_k$, for $k \geq 1$ and $\lambda_1 = 42$. Then:

– From $\mathcal{A}^3_{n-1}(0, \ldots, 0)$ we obtain $\mathcal{A}^3_n(0, \ldots, 0, t_{\frac{n-4}{2}})$, $t_{\frac{n-4}{2}} \in \mathbb{C}$.

  Hence, we can distinguish:

  * If $t_{\frac{n-4}{2}} = 0$, then
  $$\mathcal{A}^3_n(0, \ldots, 0).$$
  * If $t_{\frac{n-4}{2}} \neq 0$, then all algebras $\mathcal{A}^3_n(0, \ldots, 0, t_{\frac{n-4}{2}})$ are isomorphic to:
  $$\mathcal{A}^3_n(0, \ldots, 0, 1).$$

  And $(0, \ldots, 0)$, $(0, \ldots, 1)$ satisfy Jacobi equations (3).

– From the three algebras $\mathcal{A}^3_{n-1}(1, 0, \ldots, 0)$, $\mathcal{A}^3_{n-1}(0, \ldots, 0, 1)$ and $\mathcal{A}^3_{n-1}(0, \ldots, 0, 1, \frac{(n-4)(n-8)}{24})$ we now obtain

$$
\mathcal{A}^3_n(1, 0, \ldots, 0, \lambda), \quad \mathcal{A}^3_n(0, \ldots, 0, 1, \lambda),
$$
$$
\text{and} \quad \mathcal{A}^3_n\left(0, \ldots, 0, 1, \frac{(n-4)(n-8)}{24}, \frac{(n-6)^2(n-4)(n-8)}{32 \cdot 4 \cdot 3}\right),
$$

  respectively.

– From $\mathcal{A}^3_{n-1}(\lambda_1, \dots, \lambda_{\frac{n-6}{2}})$, where $\lambda_{k+1} = \frac{k+1}{2(2k+5)}\lambda_k$ for $k \geq 1$ and $\lambda_1 = 42$, we obtain $\mathcal{A}^3_n(\lambda_1, \dots, \lambda_{\frac{n-4}{2}})$, where $\lambda_{k+1} = \frac{k+1}{2(2k+5)}\lambda_k$ for $k \geq 1$ and $\lambda_1 = 42$, since similarly to the case n odd, it is a quotient algebra of the infinite Witt Lie algebra. $\square$

6 Rigid solvable Lie algebras whose nil-radical is 3-graded filiform

Let M^n be the algebraic variety of Lie algebras of dimension n imbedded in $\mathbb{C}^{\frac{n^3-n^2}{2}}$. A Lie algebra $\mathcal{L}$ of dimension n is called rigid if its orbit is a Zariski open set of M^n. For a general overview of these algebras, the reader can consult [5].

According to a result by Carles [2], it follows that every solvable rigid Lie algebra $\mathcal{L}$ is decomposable in the sense:

$$\mathcal{L} = \mathcal{N} \oplus \mathcal{T}$$

where $\mathcal{N}$ is the nil-radical of $\mathcal{L}$ and $\mathcal{T}$ is an exterior torus of derivations.

In [1] authors study solvable rigid Lie algebras whose nil-radical is filiform. Particularly, they study algebras whose nil-radical is the c-graded Lie algebra $\mathcal{A}^c_n(t_1, \dots, t_{[\frac{n-c+1}{2}]-1})$.

Then, by using Theorem 3.13 of [1] and our 3-graded filiform Lie algebras classification, we deduce the following:

Theorem 6.1. *Let $\mathcal{L} = \mathcal{T} \oplus \mathcal{A}^3_n(t_1, \dots, t_{[\frac{n-c+1}{2}]-1})$ be a decomposable Lie algebra of dimension $n+1$, where $\mathcal{A}^3_n(t_1, \dots, t_{[\frac{n-c+1}{2}]-1})$ is a 3-graded filiform Lie algebra. Then:*

- *$\mathcal{L}$ is not rigid if n is odd and $11 \leq n \leq 13$ or if n is even and $12 \leq n \leq 14$.*
- *$\mathcal{L}$ is rigid if $n \geq 15$ with n odd or $n \geq 16$ with n even, and $\mathcal{A}^3_n$ is neither the model algebra nor an algebra belonging to the families $\mathcal{A}^3_n(1, 0, \dots, 0, \lambda)$ and $\mathcal{A}^3_n(0, \dots, 1, \lambda)$.* $\square$

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